第八章　重积分

8.3　三重积分的计算法

直角坐标系下，计算三重积分的方法也是将它化为累次积分,即化为先定积分后二重积分或先二重积分后定积分的形式,从而化为三次积分,这两种方法称为”投影”法和”切片”法。

三重积分的物理意义：空间立体的质量可以通过密度函数的三重积分来表示，即

$$m=\iiint_{\Omega}\rho(x,y,z)dv=\lim_{\lambda\to 0}\sum_{i=1}^{n}\rho(\xi_{\mathrm{i}},\eta_i,\xi_i)\Delta v_i$$

8.3.1 三重积分的定义

定义8.3.1 $$\iiint_{\Omega} f(x,y,z)dv = \lim_{\lambda\to 0}\sum_{i=1}^{n} f(\xi_i,\eta_i,\zeta_i)\Delta v_i$$

其中dv叫做体积元素。

在直角坐标系中，如果用平行于坐标面的平面来划分Ω，那末除了包含Ω的边界点的一些不规则小闭区域外，得到的小闭区域Δv_i为长方体。

设长方体小闭区域Δv_i的边长为Δx_j，Δy_k，Δz_l，则$\Delta v_i = \Delta x_j \Delta y_k \Delta z_l$。

在直角坐标系下的体积元素：$dv=dxdydz$

$$\iiint_{\Omega} f(x,y,z)dv = \iiint_{\Omega} f(x,y,z)dxdydz$$

8.3.2 直角坐标系下的三重积分的计算法

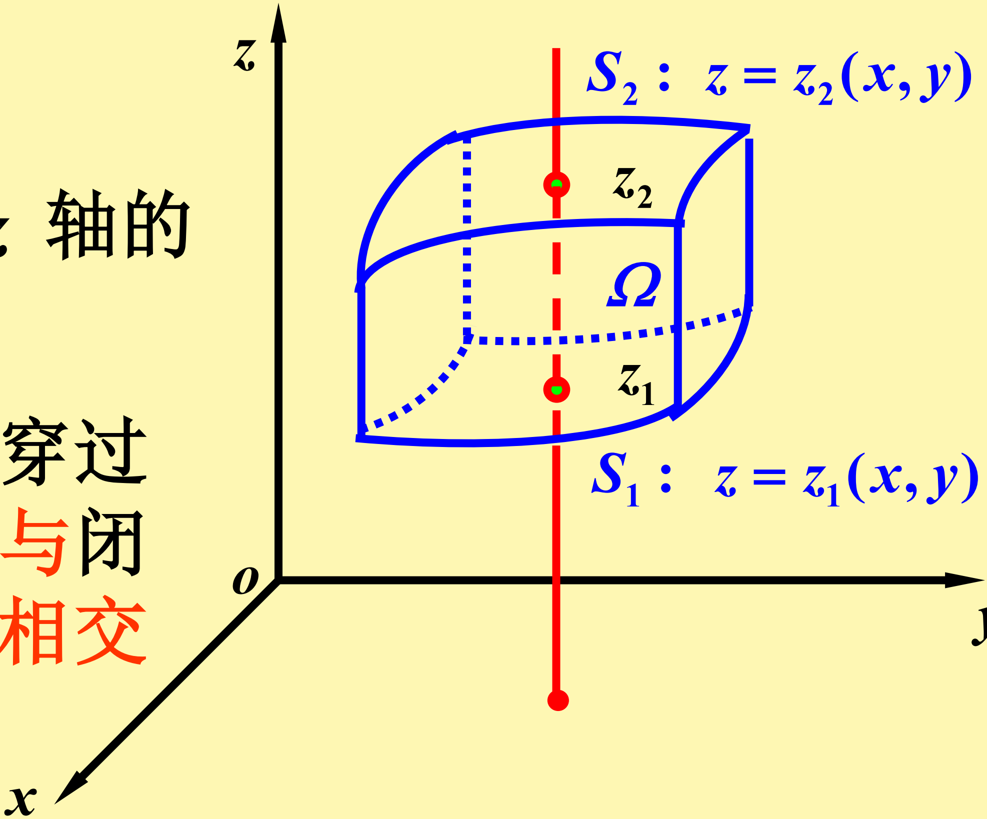
基本方法：化三重积分为三次单积分

$$dv=dxdydz$$

一、投影法

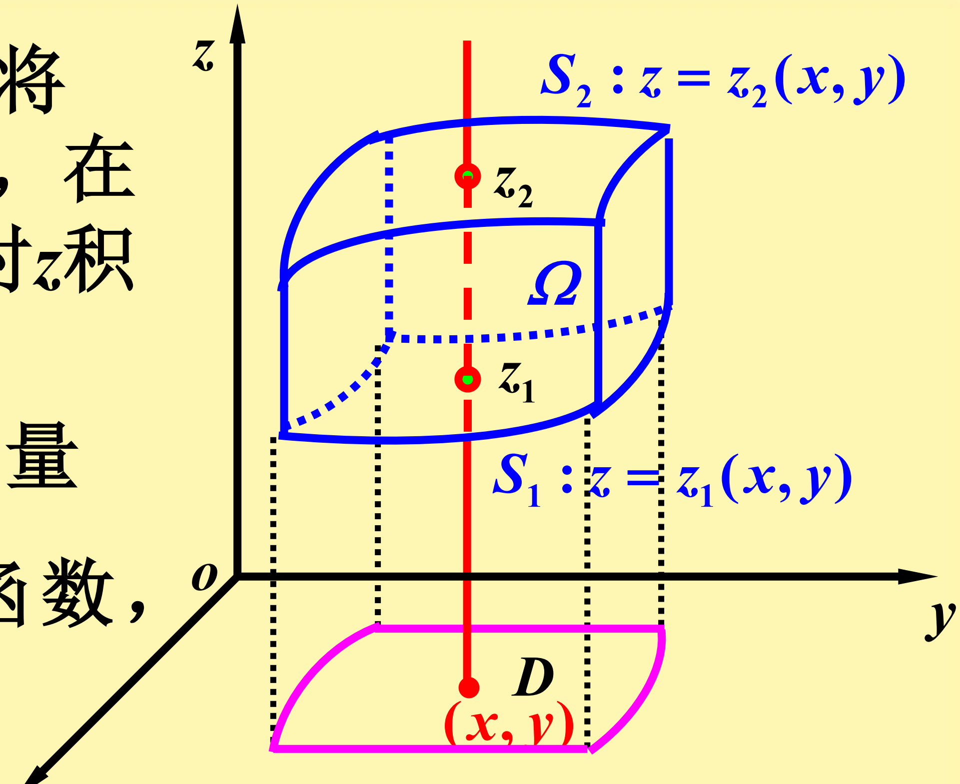
1、 Ω为母线平行于z 轴的柱体时

假设平行于z轴且穿过闭区域Ω内部的直线与闭区域Ω的边界曲面S相交不多于两点。



先将x，y看作定值，将$f\ (x,y,z)$只看作z的函数，在区间$[z_1(x,y),\ z_2(x,y)]$上对z积分。

物理意义：细直棒的质量



积分的结果是x,y的函数，记为$F(x,y)$，即

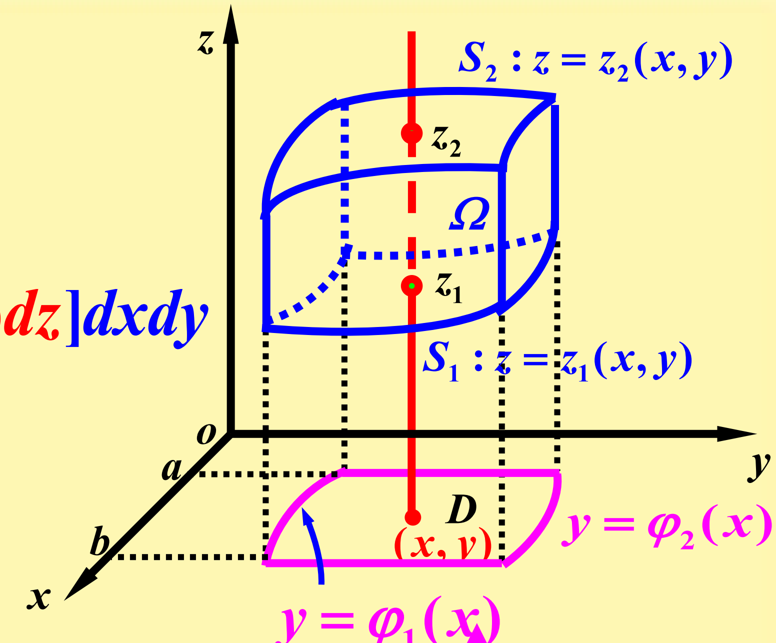
$$F(x,y) = \int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)dz$$

然后计算$F(x,y)$在闭区域D的二重积分

$$\iint_D F(x,y)d\sigma = \iint_D [\int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)dz]d\sigma \quad 。$$

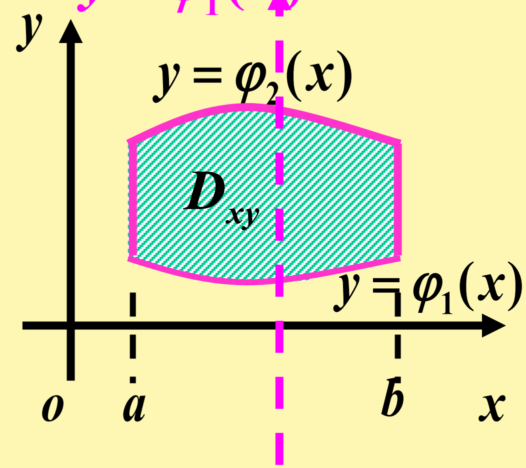
若Ω在xoy平面上的投影区域记为D_{xy}，则有

$$\iiint_{\Omega} f(x,y,z)dv=\iint_{D_{xy}}\left[\int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)dz\right]dxdy$$



投影区域D_{xy}用不等式表示:

$$\varphi_1(x)\le y\le\varphi_2(x),\quad a\le x\le b$$



则将二重积分化为二次积分，得到三重积分的计算公式:

$$\iiint_{\Omega} f(x,y,z)=\int_a^b dx\int_{\varphi_1(x)}^{\varphi_2(x)}\left[\int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)dz\right]dy$$

$$\iiint_{\Omega} f(x,y,z)dv = \iint_{D_{xy}}\left[\int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)dz\right]dxdy$$

$$= \int_a^b dx\int_{\varphi_1(x)}^{\varphi_2(x)} dy\int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)dz$$

把三重积分化为先对z、次对y、最后对x的三次积分

若 D_{xy}： $\psi_1(y)\le x\le \psi_2(y), c\le y\le d$

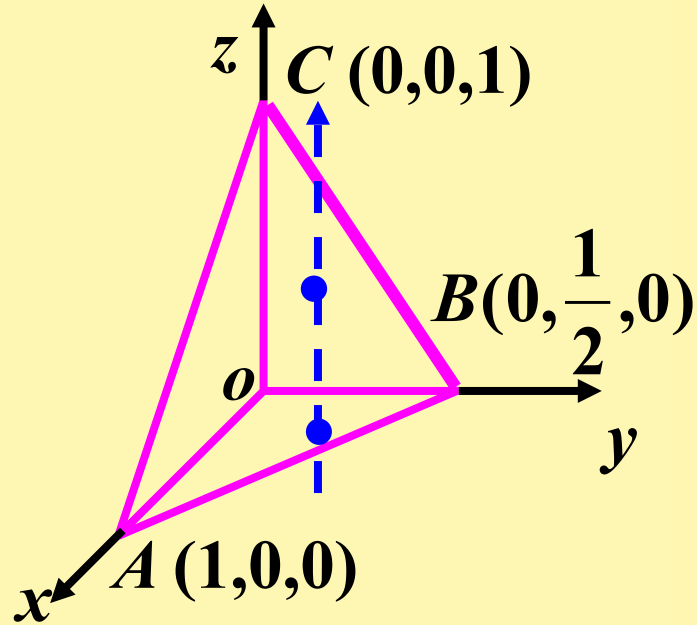
$$\iiint_{\Omega} f(x,y,z) = \int_c^d dy\int_{\psi_1(y)}^{\psi_2(y)} dx\int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)dz$$

上式的数学方法概括为：

“先单后重法”，或“投影法”

例1 计算三重积分 $\iiint\limits_{\Omega} x\,dx\,dy\,dz$，

其中 Ω 为三个坐标面及平面 $x+2y+z=1$ 所围成的闭区域



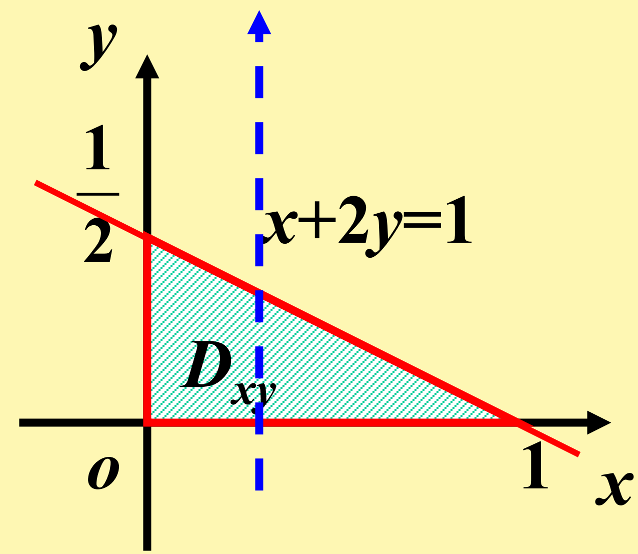
解 $\iiint\limits_{\Omega} x\,dx\,dy\,dz$

$$= \iint\limits_{D_{xy}} \left(\int_0^{1-x-2y} x\,dz \right) d\sigma$$

$$= \int_0^1 dx \int_0^{\frac{1-x}{2}} dy \int_0^{1-x-2y} x\,dz$$

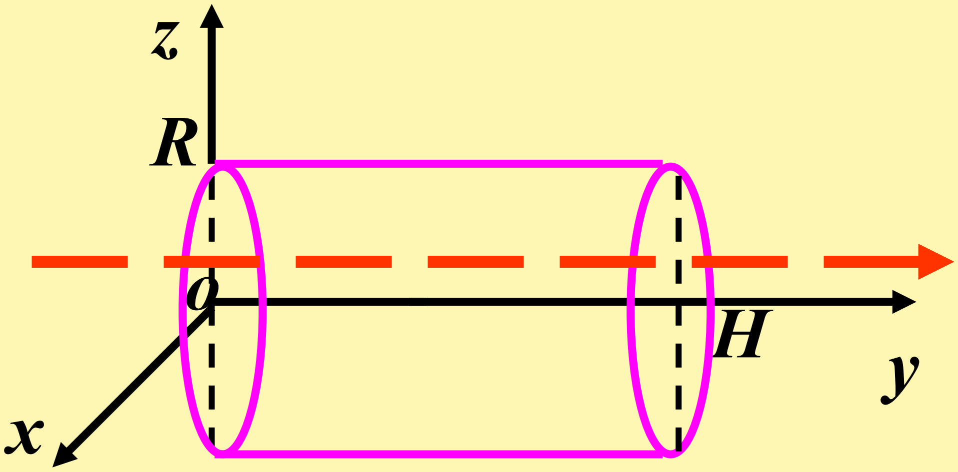
$$= \int_0^1 dx \int_0^{\frac{1-x}{2}} x(1-x-2y)\,dy$$

$$= \frac{1}{4}\int_0^1 (x-2x^2+x^3)\,dx = \frac{1}{48}。$$



2、 Ω为母线平行y轴或 x 轴的柱体时

(1) $\Omega: y_1(z,x) \le y \le y_2(z,x), (z,x) \in D_{zx}$

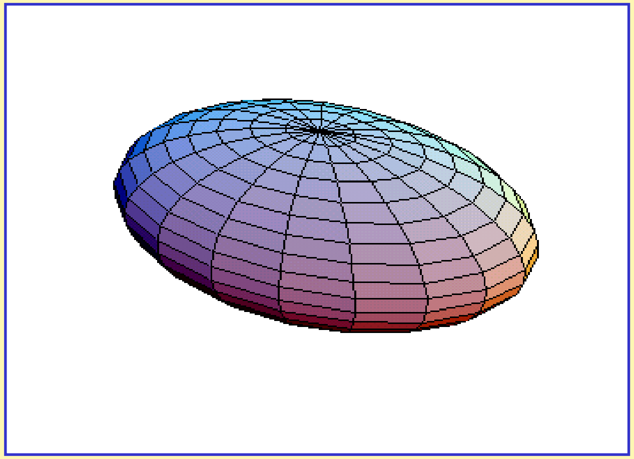


$$\iiint_{\Omega} f(x,y,z)dv = \iint_{D_{zx}} dzdx \int_{y_1(z,x)}^{y_2(z,x)} f(x,y,z)dy$$

(2) $\Omega: x_1(y,z) \le x \le x_2(y,z), (y,z) \in D_{yz}$

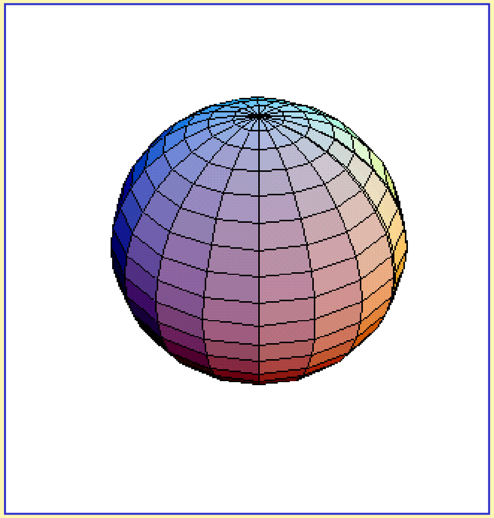
$$\iiint_{\Omega} f(x,y,z)dv = \iint_{D_{yz}} dydz \int_{x_1(y,z)}^{x_2(y,z)} f(x,y,z)dx$$

常见曲面：



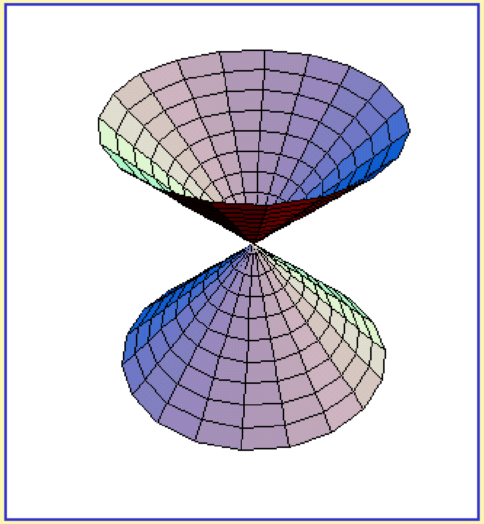
（1）椭球面

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$



（2）球面

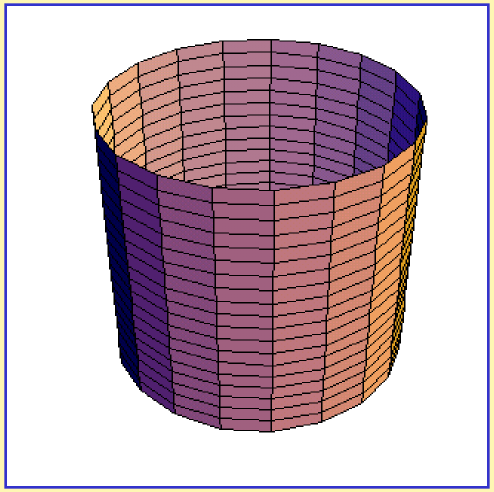
$$x^2+y^2+z^2=a^2$$

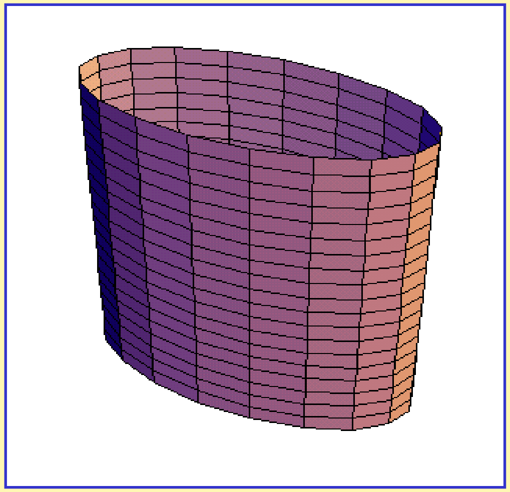


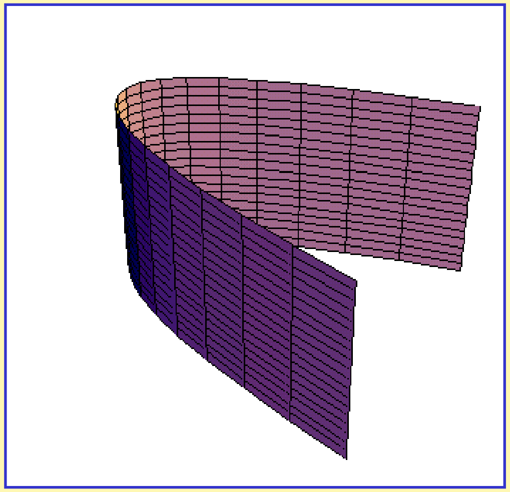
（3）圆锥面

$$x^2+y^2=z^2$$

$$z=\pm\sqrt{x^2+y^2}$$







(4) 圆柱面

$$x^2 + y^2 = R^2$$

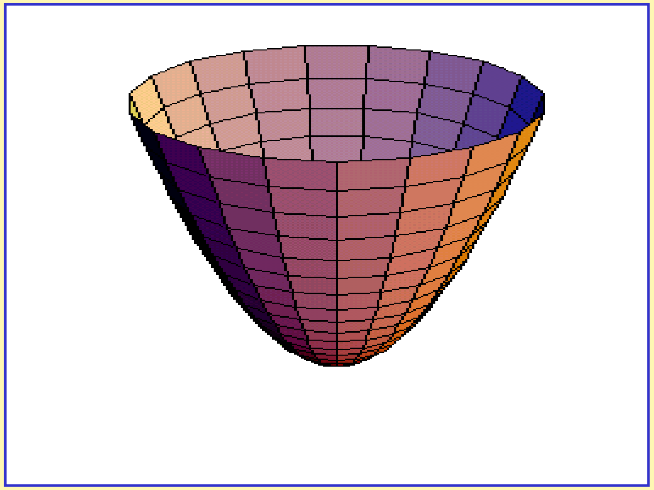
(5) 椭圆柱面

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

(6) 抛物柱面

$$x^2 = 2py$$

$$(p > 0)$$

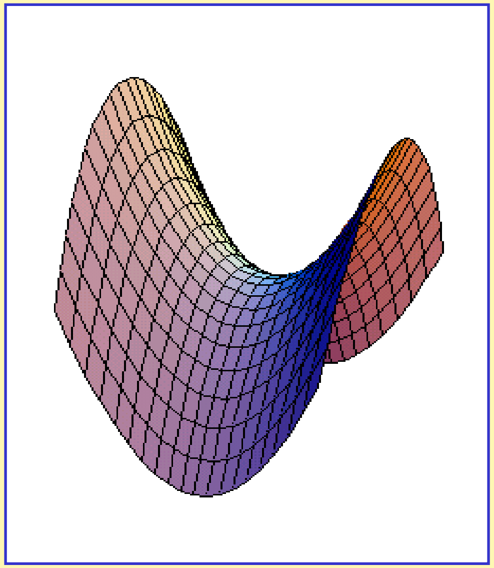


（7）椭圆抛物面

$$\frac{x^2}{2p}+\frac{y^2}{2q}=z$$

(p 与 q 同号)

$$x^2+y^2=z$$



（8）马鞍面

$$-\frac{x^2}{2p}+\frac{y^2}{2q}=z$$

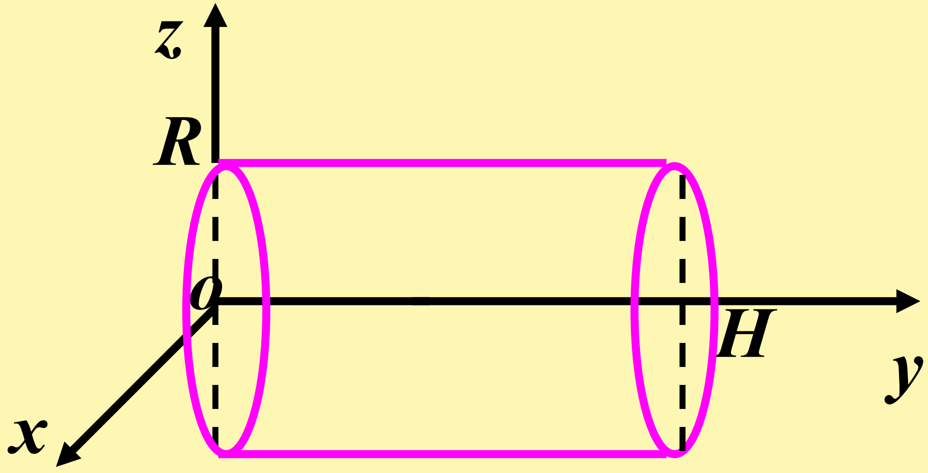
(p 与 q 同号)

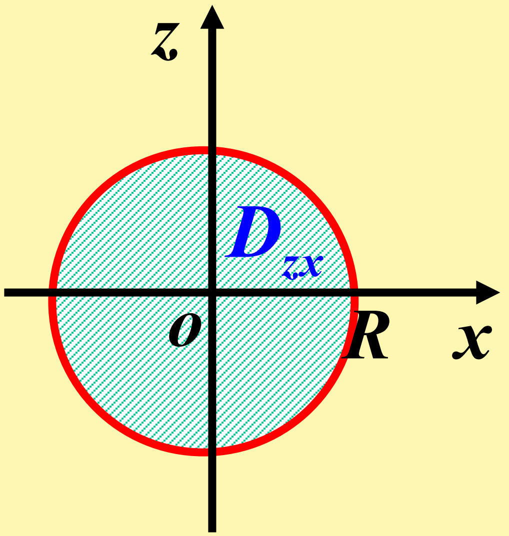
$$z=xy$$

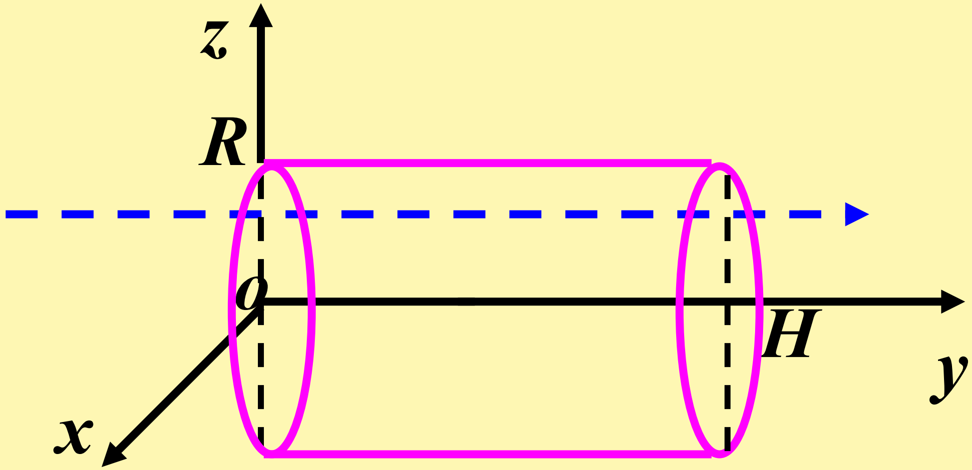
例2 将下列三重积分 $\iiint\limits_{\Omega} f(x,y,z)dv$ 化为三次积分形式,其中Ω为

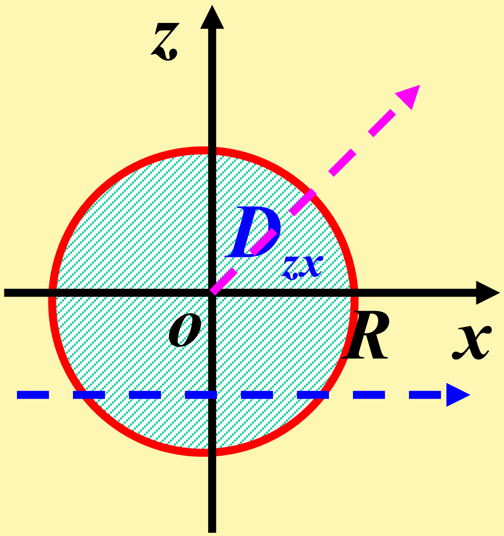
$$(1)\Omega: x^2+z^2=R^2, y=0, y=H;$$

解 (1)Ω及在zox面上的投影如下图









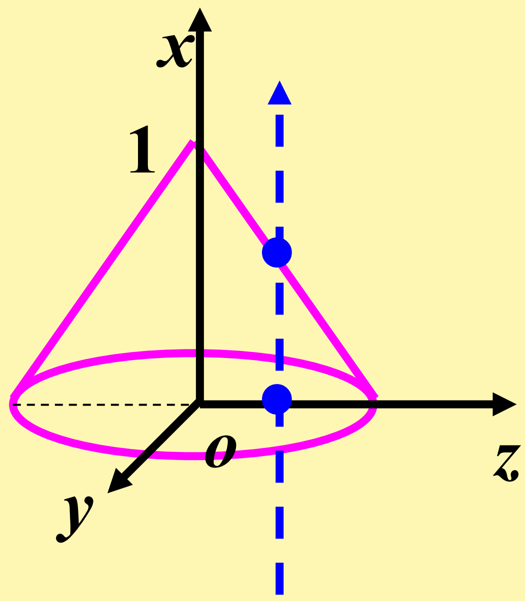
$$\iiint_{\Omega} f(x,y,z)dv = \iint_{D_{zx}} \left(\int_0^H f(x,y,z)\,dy\right)d\sigma$$

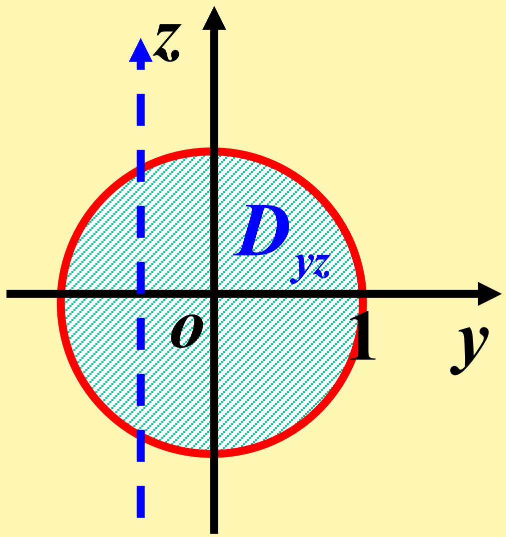
$$= \iint_{D_{zx}} d\sigma \int_0^H f(x,y,z)\,dy = \int_{-R}^{R} dz \int_{-\sqrt{R^2-z^2}}^{\sqrt{R^2-z^2}} dx \int_0^H f(x,y,z)dy$$

$$= \int_0^{2\pi} d\theta \int_0^R r dr \int_0^H f(r\cos\theta, y, r\sin\theta)dy$$

$(2)\Omega: x = 1-\sqrt{y^2+z^2}, x = 0$

Ω及在yoz面上的投影如下图





$$\iiint_{\Omega} f(x,y,z)dv = \iint_{D_{yz}} \left(\int_0^{1-\sqrt{y^2+z^2}} f(x,y,z)\,dx\right)dydz$$

$$= \int_{-1}^{1} dy \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} dz \int_0^{1-\sqrt{y^2+z^2}} f(x,y,z)dx。$$

例3 计算下列三重积分

(1)$\iiint\limits_{\Omega} \dfrac{z\ln(x^2+y^2+z^2+1)}{x^2+y^2+z^2+1}dv$,其中$\Omega$:

$x^2+y^2+z^2 \le 1$ 。

被积函数关于z是奇函数，关于x, y是偶函数

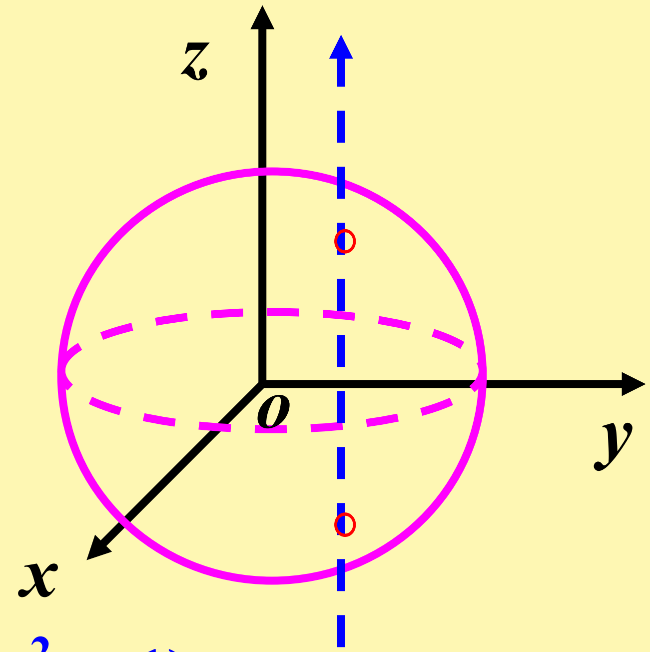
(2)$\iiint\limits_{\Omega} x^2yzdv$,其中$\Omega$ 是由曲面$z=x^2+y^2$

与$z=H \ \ (H>0)$所围成的区域。

x^2yz关于 y, z是奇函数,关于 x是偶函数

$$(1)\iiint_{\Omega}\frac{z\ln(x^2+y^2+z^2+1)}{x^2+y^2+z^2+1}dv$$

$$\Omega: x^2+y^2+z^2\le 1$$



$$\iiint_{\Omega}\frac{z\ln(x^2+y^2+z^2+1)}{x^2+y^2+z^2+1}dv$$

$$=\iint_{D_{xy}}dxdy\int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}}\frac{z\ln(x^2+y^2+z^2+1)}{x^2+y^2+z^2+1}dz$$

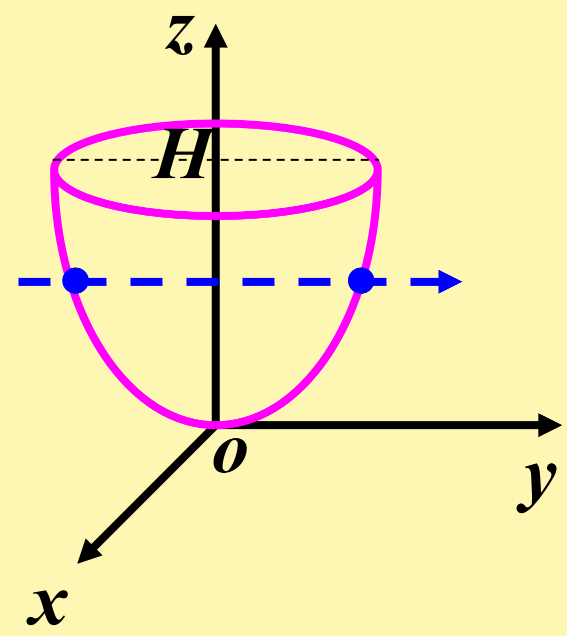
$$=\iint_{D_{xy}}0\ dxdy=0。$$

区域Ω关于xoy面对称

被积函数是关于z是奇函数

$$(2)\iiint_{\Omega} x^2 yz\,dv$$

$$\Omega: x^2 + y^2 \le z \le H\ (H > 0)$$



解
$$\iiint_{\Omega} x^2 yz\,dv$$

$$= \iint_{D_{zx}} dxdz \int_{-\sqrt{z-x^2}}^{\sqrt{z-x^2}} x^2 yz\,dy$$

$$= \iint_{D_{zx}} x^2 z \cdot 0\,dxdz = 0。$$

$f(x, y, z)$关于y是奇函数

区域Ω是关于zox面是对称的

Ω	$f(x,y,z)$	$\iiint_{\Omega} f(x,y,z)dv$
关于 xOy 平面对称	关于z是奇函数	0
	关于z是偶函数	$2\iiint_{\Omega_1} f(x,y,z)dv$
关于 yOz 平面对称	关于x是奇函数	0
	关于x是偶函数	$2\iiint_{\Omega_1} f(x,y,z)dv$
关于 xOz 平面对称	关于y是奇函数	0
	关于y是偶函数	$2\iiint_{\Omega_1} f(x,y,z)dv$

二、奇偶函数在对称区域上的积分性质

1、若空间区域Ω是关于zox面是对称的,则

$$\iiint_{\Omega} f(x,y,z)dv=\iiint_{\Omega_1}\big(f(x,y,z)+f(x,-y,z)\big)dv$$

$$=\begin{cases} 0 & f\text{关于}y\text{是奇函数} \\ 2\iiint_{\Omega_1} f(x,y,z)dv & f\text{关于}y\text{是偶函数} \end{cases}$$

其中Ω_1是Ω的右半部分

f关于y是奇函数：$f(x,y,z)=-f(x,-y,z)$

2、若空间区域Ω是关于yoz面是对称的,则

$$\iiint_{\Omega} f(x,y,z)dv=\iiint_{\Omega_1}\big(f(x,y,z)+f(-x,y,z)\big)dv$$

$$=\begin{cases} 0 & f\text{关于}x\text{是奇函数} \\ 2\iiint_{\Omega_1} f(x,y,z)dv & f\text{关于}x\text{是偶函数} \end{cases}$$

其中Ω_1是Ω的前半部分

f关于x是奇函数：$f(x,y,z)=-f(-x,y,z)$

3、若空间区域Ω是关于xoy面是对称的,则

$$\iiint_{\Omega} f(x,y,z)dv=\iiint_{\Omega_1}\Big(f(x,y,z)+f(x,y,-z)\Big)dv$$

$$=\begin{cases} 0 & f\text{关于}z\text{是奇函数} \\ 2\iiint_{\Omega_1} f(x,y,z)dv & f\text{关于}z\text{是偶函数} \end{cases}$$

其中Ω_1是Ω的上半部分

若积分区域Ω关于坐标面对称，则：

$$\iiint_{\Omega} f(x,y,z)dv=\iiint_{\Omega_1}\Big(f(x,y,z)+f(\text{对称点})\Big)dv$$

Ω_1是Ω的靠近第一卦限的部分

4、若空间区域Ω： $(a,b,c)\in\Omega\Rightarrow(b,a,c)\in\Omega$

则：$\iiint\limits_{\Omega} f(x,y,z)dv=\iiint\limits_{\Omega} f(y,x,z)dv$

如：$f(x,y,z)=xy^2+z \quad \Omega: x^2+y^2\le z\le 1$

则 $\iiint\limits_{\Omega}\left(xy^2+z\right)dv=\iiint\limits_{\Omega}\left(yx^2+z\right)dv$

若区域Ω中点的坐标具有轮换对称性，即：

$(a,b,c)\in\Omega\Rightarrow(b,c,a),(c,a,b)\in\Omega$

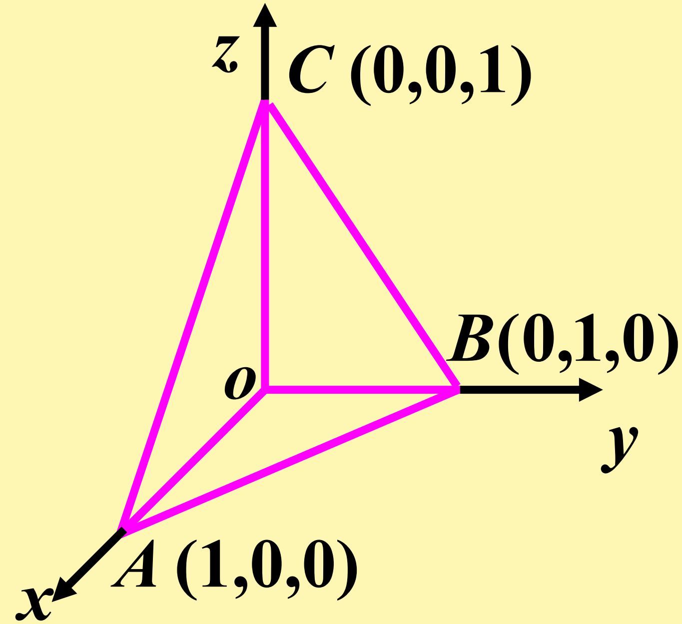
则：$\iiint\limits_{\Omega} f(x,y,z)dv=\iiint\limits_{\Omega} f(y,z,x)dv=\iiint\limits_{\Omega} f(z,x,y)dv$

注意：利用对称性时关键是看区域的特征

例3 (3) $\iiint\limits_{\Omega}(x+y+z)dv$,其中$\Omega$是由曲面

$x+y+z=1$和三个坐标面所围成的区域。

解 空间区域Ω如图所示:



由于空间区域Ω对三个变量是对称的,因此有:

$$\iiint\limits_{\Omega} xdv = \iiint\limits_{\Omega} ydv = \iiint\limits_{\Omega} zdv$$

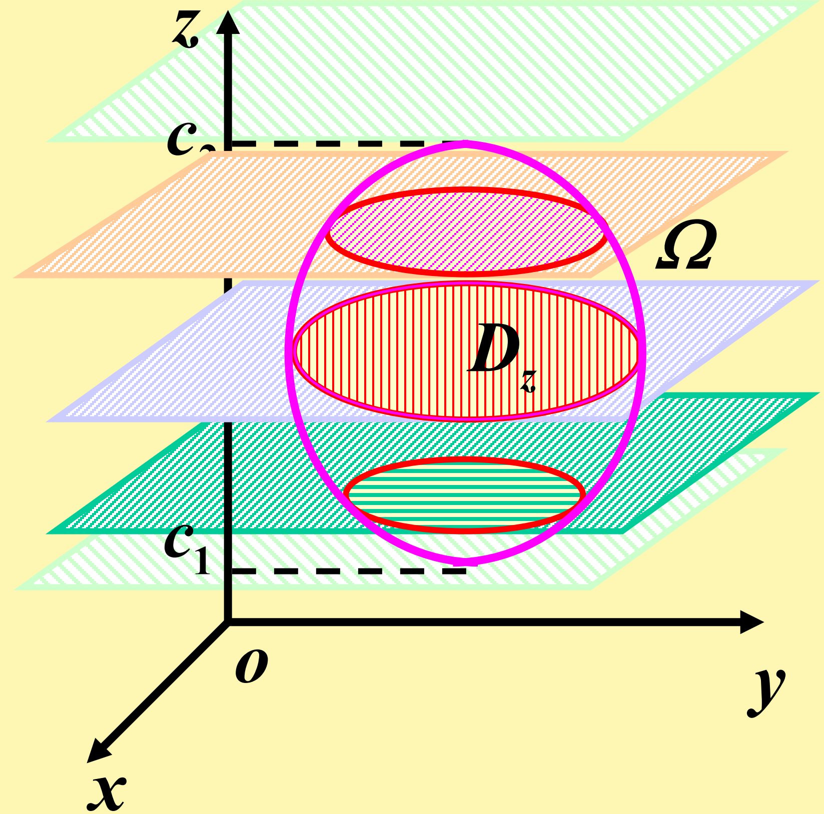
$$\therefore \iiint\limits_{\Omega}(x+y+z)dv = 3\iiint\limits_{\Omega} xdv = \cdots$$

$$= 3\cdot\frac{1}{24} = \frac{1}{8}$$

三、切片法

又叫“先重后单法”

设区域Ω夹在平面 $z=c_1$，$z=c_2(c_1<c_2)$之间

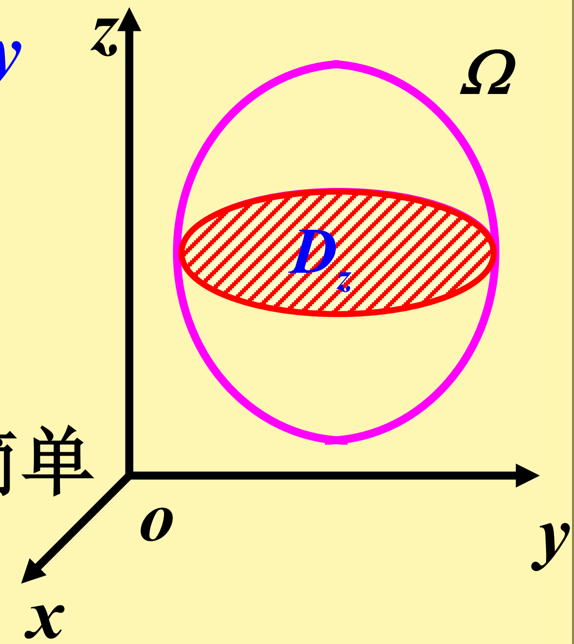


用竖坐标为z $(c_1\le z\le c_2)$的平面截Ω所得截面为D_z或$D(z)$，即

$$\Omega=\{(x,y,z)\,|\,(x,y)\in D_z, c_1\le z\le c_2\}$$

$$\iiint_{\Omega} f(x,y,z)dv=\int_{c_1}^{c_2}dz\iint_{D_z} f(x,y,z)dxdy \qquad (3)$$

$$\iiint_{\Omega} f(x,y,z)dv = \int_{c_1}^{c_2} dz \iint_{D_z} f(x,y,z)dxdy$$



上式的适用范围：

①D_z简单（圆、椭圆、长方形等）

② $f(x,y,z)$ 在D_z上对x、y的二重积分简单

特别当 $f(x,y,z)$ 只是 z 的函数：

$$\iiint_{\Omega} f(x,y,z)dv = \int_{c_1}^{c_2} dz \iint_{D_z} \varphi(z)dxdy = \int_{c_1}^{c_2} \varphi(z)\sigma_{D_z} dz$$

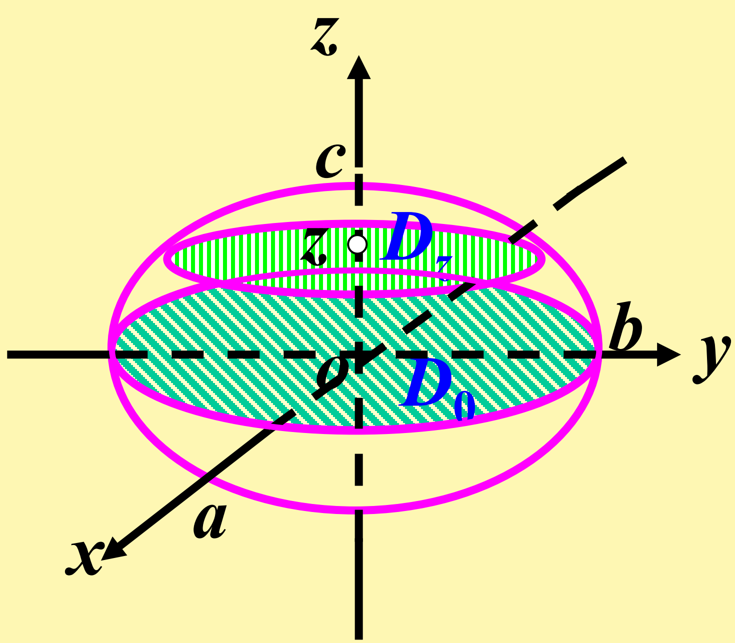
类似地 $$\iiint_{\Omega} f(x,y,z)dv = \int_{a_1}^{a_2} dx \iint_{D_x} f(x,y,z)dydz$$

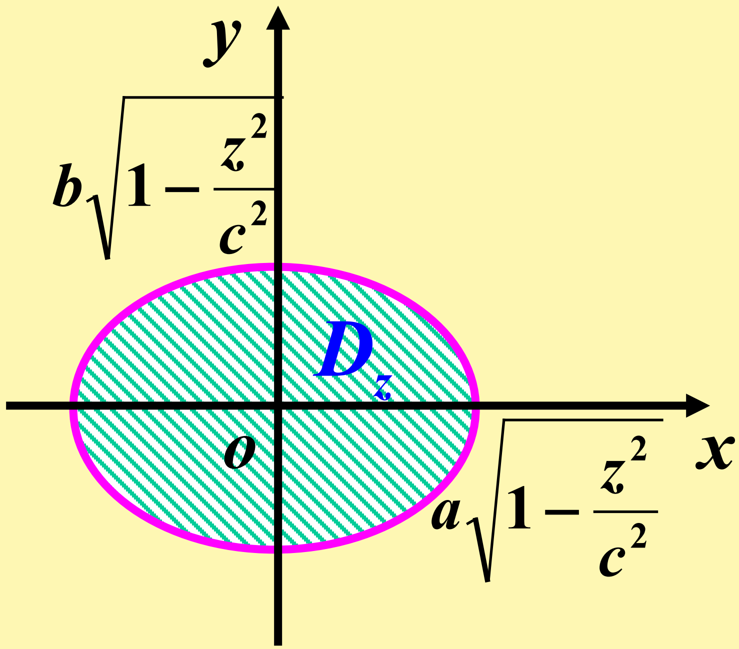
$$\iiint_{\Omega} f(x,y,z)dv = \int_{b_1}^{b_2} dy \iint_{D_y} f(x,y,z)dzdx$$

例4 计算三重积分$\iiint\limits_{\Omega} z^2 dv$，其中$\Omega$是由椭球面

$\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$所围成的空间闭区域。

解



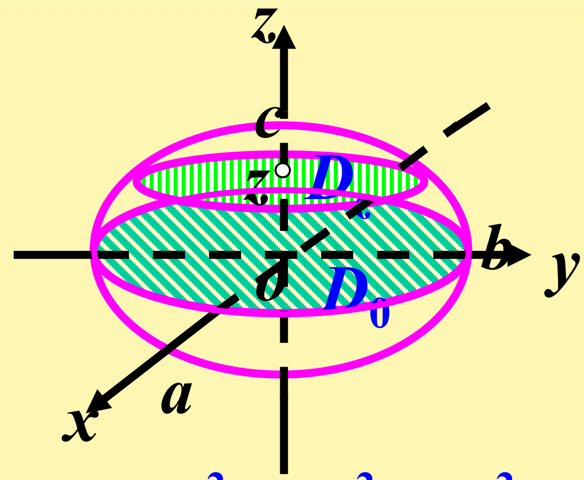


$\Omega: -c \le z \le c, (x, y) \in D_z$，　$D_z: \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2} \le 1-\dfrac{z^2}{c^2}$。

用“切片法”较方便

①D_z是椭圆域，较简单

②$f(x,y,z)=z^2$只是 z 的函数



$$\Omega: \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\le 1$$

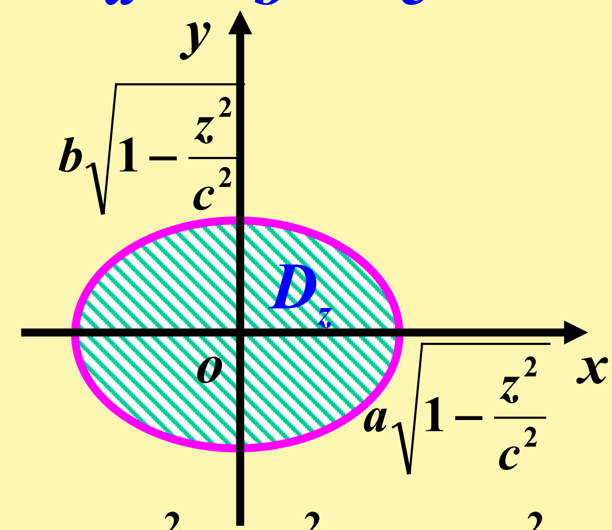
$$\iiint_{\Omega} z^2 dxdydz$$

$$=\int_{-c}^{c} z^2 dz \iint_{D_z} dxdy$$

$$=\int_{-c}^{c} z^2 \sigma(D_z) dz$$

$$=\pi ab\int_{-c}^{c}\left(1-\frac{z^2}{c^2}\right)z^2 dz$$

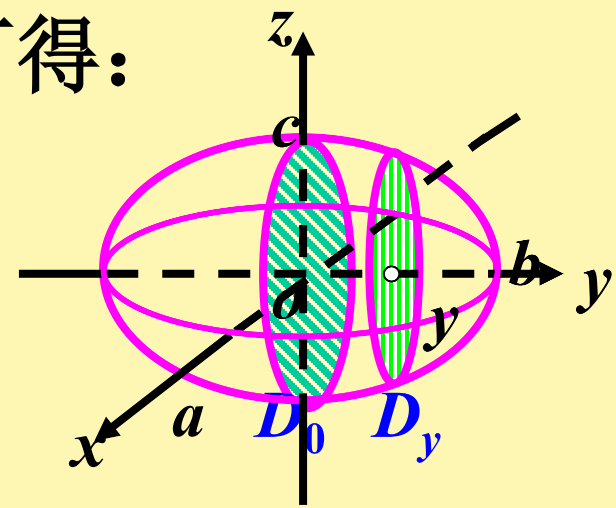
$$=\frac{4}{15}\pi abc^3$$



$$D_z: \frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1-\frac{z^2}{c^2}。$$

$$\iiint_{\Omega} z^2 dxdydz = \frac{4}{15}\pi abc^3$$

类似可得：



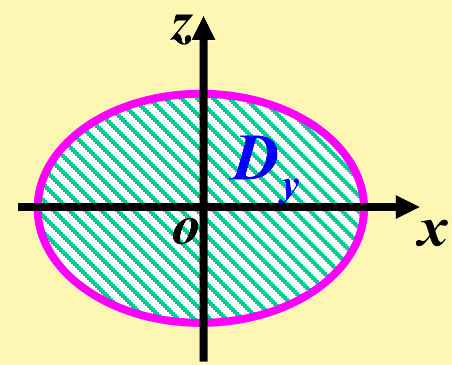
$$\iiint_{\Omega} y^2 dxdydz = \int_{-b}^{b} y^2 dy \iint_{D_y} dzdx$$

$$= \int_{-b}^{b} y^2 \pi ac(1-\frac{y^2}{b^2})dy = \frac{4}{15}\pi ab^3 c。$$

$$\Omega : \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\le 1$$

$$\iiint_{\Omega} x^2 dxdydz = \int_{-a}^{a} x^2 dx \iint_{D_x} dydz$$

$$= \int_{-a}^{a} x^2 \pi bc(1-\frac{x^2}{a^2})dx = \frac{4}{15}\pi a^3 bc;$$



$$\iiint_{\Omega} (x^2+y^2+z^2)dxdydz$$

$$= \frac{4}{15}\pi abc(a^2+b^2+c^2)$$

$$D_y : \frac{x^2}{a^2}+\frac{z^2}{c^2}\le 1-\frac{y^2}{b^2}$$

练习一 计算三重积分$\iiint\limits_{\Omega} z\,dv$

$\Omega: \sqrt{x^2+y^2} \le z \le 1$。

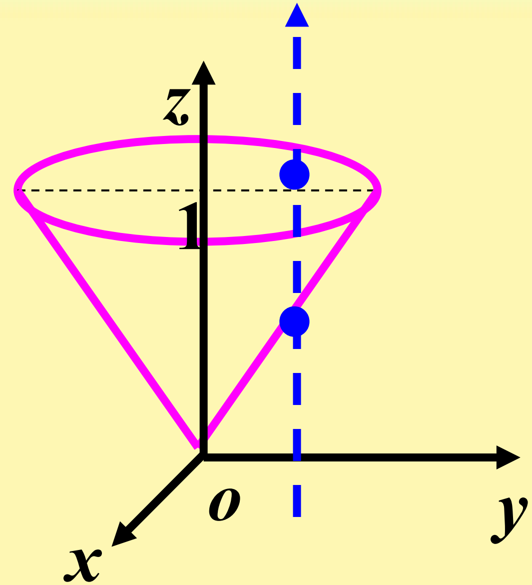
解法一 用“先单后重法”

$$\iiint\limits_{\Omega} z\,dv = \iint\limits_{D_{xy}} \left(\int_{\sqrt{x^2+y^2}}^{1} z\,dz\right) d\sigma$$

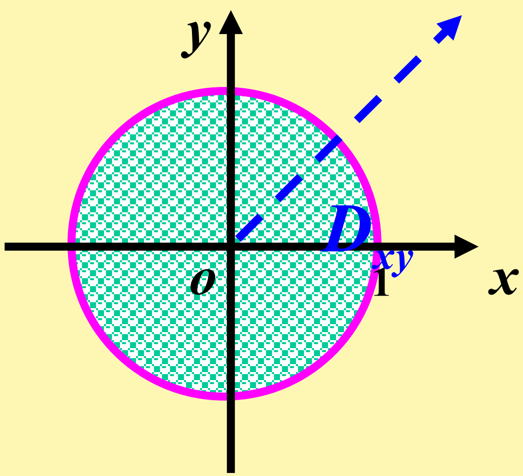
$$= \iint\limits_{D_{xy}} \frac{z^2}{2}\Bigg|_{\sqrt{x^2+y^2}}^{1} d\sigma$$

$$= \frac{1}{2}\iint\limits_{D_{xy}} [1-(x^2+y^2)]\,d\sigma$$

$$= \frac{1}{2}\int_0^{2\pi} d\theta \int_0^1 (1-r^2)\,r\,dr = \frac{\pi}{4}$$



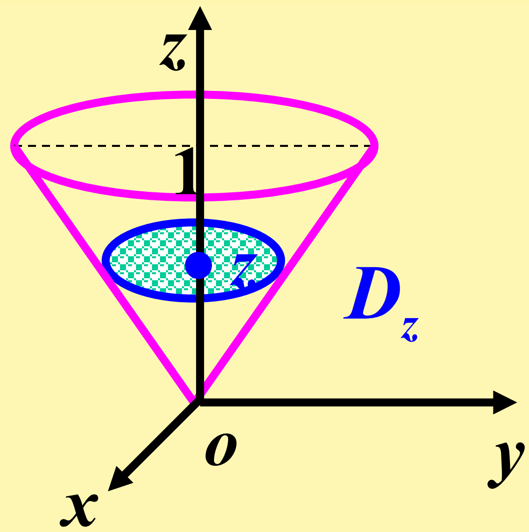
$\Omega: \sqrt{x^2+y^2} \le z \le 1$



D_{xy}: $x^2+y^2 \le 1$

练习一 计算三重积分$\iiint\limits_{\Omega} z dv$

$\Omega: \sqrt{x^2+y^2} \le z \le 1$。



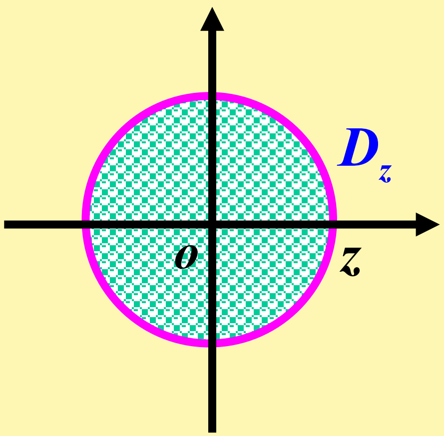
解法二 用先重后单法。

用平行于xoy面的平面去截空间区域Ω,得平面闭区域D_z,

$$\iiint\limits_{\Omega} z dv = \int_0^1 dz \iint\limits_{D_z} z\, dxdy$$

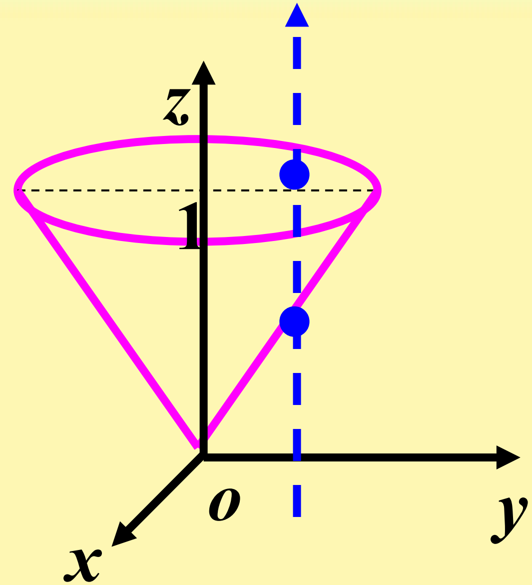
$$= \int_0^1 z dz \iint\limits_{D_z} d\sigma = \int_0^1 z(\pi z^2) dz$$

$$= \pi \int_0^1 z^3 dz = \pi \frac{z^4}{4}\Big|_0^1 = \frac{\pi}{4}。$$



$D_z: x^2+y^2 \le z^2$

计算$\iiint\limits_{\Omega} z dv$ $\Omega: \sqrt{x^2+y^2} \le z \le 1$。

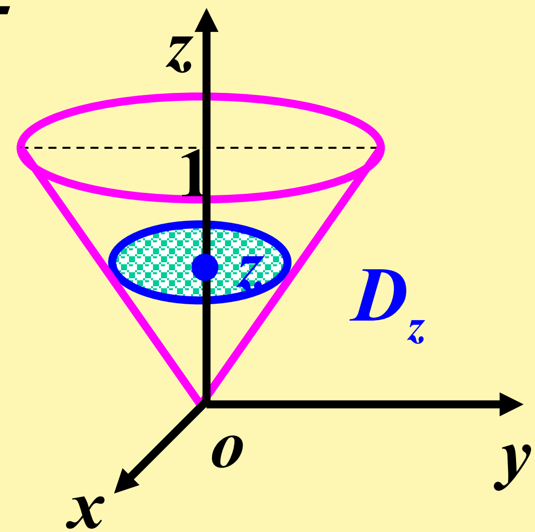


虽然被积函数关于z是奇函数，

但积分区域不关于坐标面xOy

对称，不能用对称性

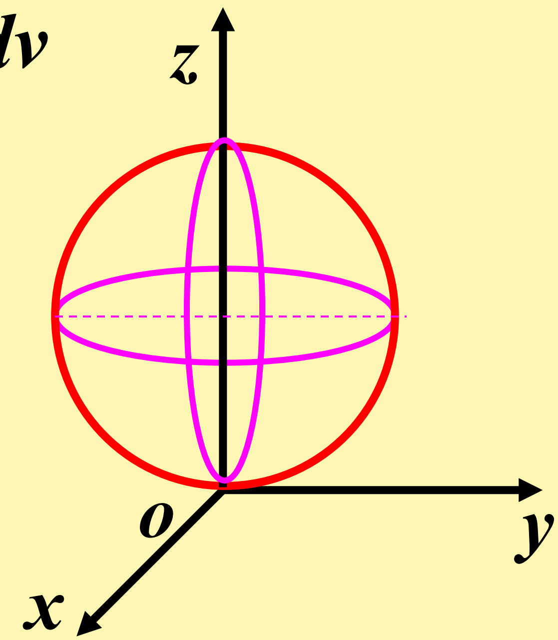
投影法：$\iiint\limits_{\Omega} z dv = \iint\limits_{D_{xy}} \left(\int_{\sqrt{x^2+y^2}}^{1} z dz\right) d\sigma$



切片法：$\iiint\limits_{\Omega} z dv = \int_0^1 dz \iint\limits_{D_z} z d\sigma$

练习二 计算 $I=\iiint\limits_{\Omega}(y^4\sin x+z)dv$

$\Omega: x^2+y^2+z^2\le 2Rz$。



解 Ω关于yoz平面对称，

$y^4\sin x$关于x是奇函数

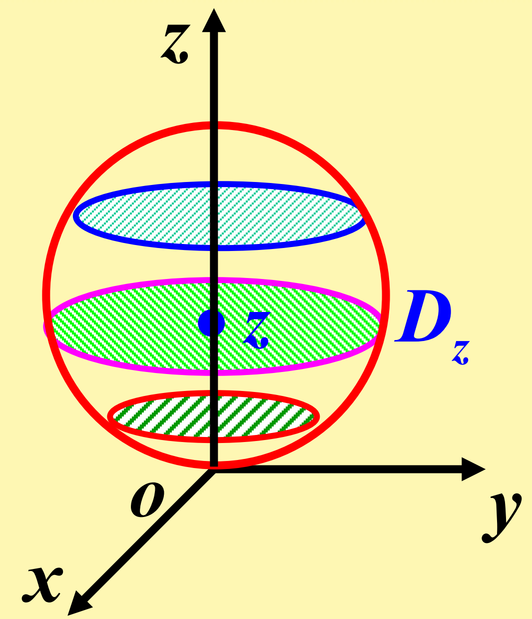
$\therefore \iiint\limits_{\Omega} y^4\sin x dv=0$

$\Omega: x^2+y^2+(z-R)^2\le R^2$

$\because I=\iiint\limits_{\Omega}(y^4\sin x+z)dv$

$=\iiint\limits_{\Omega} z dv$

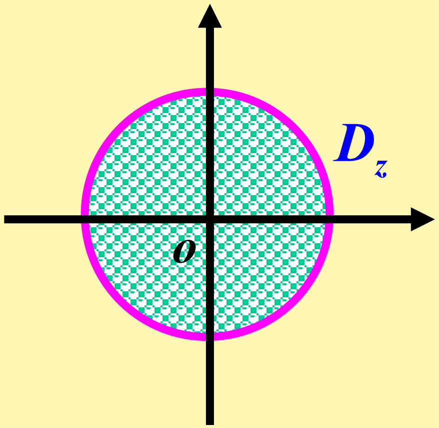
计算 $I=\iiint\limits_{\Omega}(y^4\sin x+z)dv$



$\Omega: x^2+y^2+z^2\le 2Rz$。

$\Omega:\ 0\le z\le 2R$

解法一　用先重后单法。



$$\iiint\limits_{\Omega} zdv = \int_0^{2R} zdz\iint\limits_{D_z} dxdy$$

$$=\int_0^{2R} z\sigma(z)dz$$

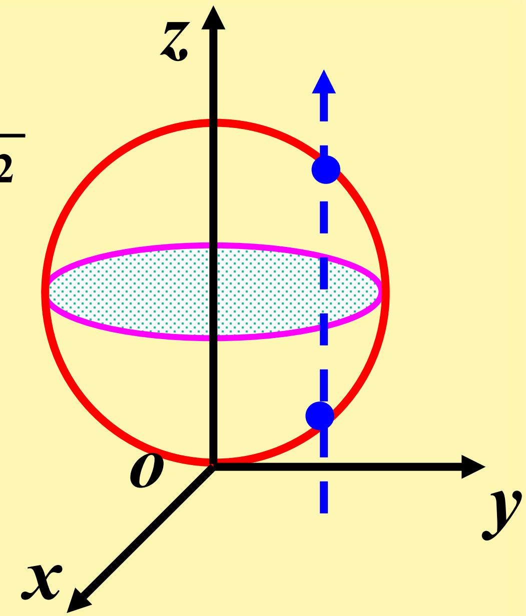
$$=\int_0^{2R}\pi(2Rz^2-z^3)dz$$

$$=\frac{4}{3}\pi R^4$$

$D_z: x^2+y^2\le 2Rz-z^2$

$$\Omega: x^2+y^2+(z-R)^2 \le R^2$$

$$R-\sqrt{R^2-x^2-y^2} \le z \le R+\sqrt{R^2-x^2-y^2}$$



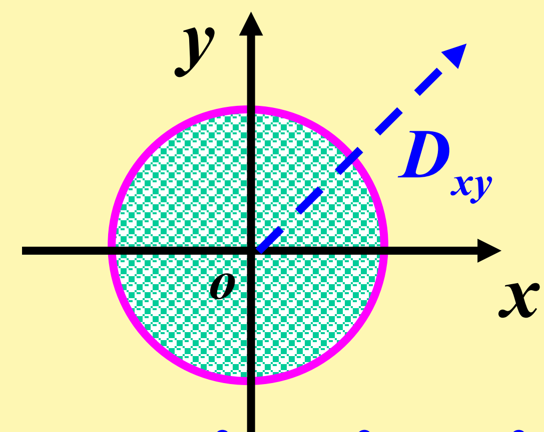
解法二　用“先单后重法”

$$\iiint\limits_{\Omega} z\,dv = \iint\limits_{D_{xy}} \left(\int_{R-\sqrt{R^2-x^2-y^2}}^{R+\sqrt{R^2-x^2-y^2}} z\,dz \right) d\sigma$$

$$= \iint\limits_{D_{xy}} \frac{z^2}{2} \Bigg|_{R-\sqrt{R^2-x^2-y^2}}^{R+\sqrt{R^2-x^2-y^2}} d\sigma$$

$$= \frac{1}{2} \iint\limits_{D_{xy}} 4R\sqrt{R^2-x^2-y^2}\,d\sigma$$

$$= \frac{1}{2} \int_0^{2\pi} d\theta \int_0^R 4Rr\sqrt{R^2-r^2}\,dr = \frac{4\pi R^4}{3}$$



$$D_{xy}: x^2+y^2 \le R^2$$

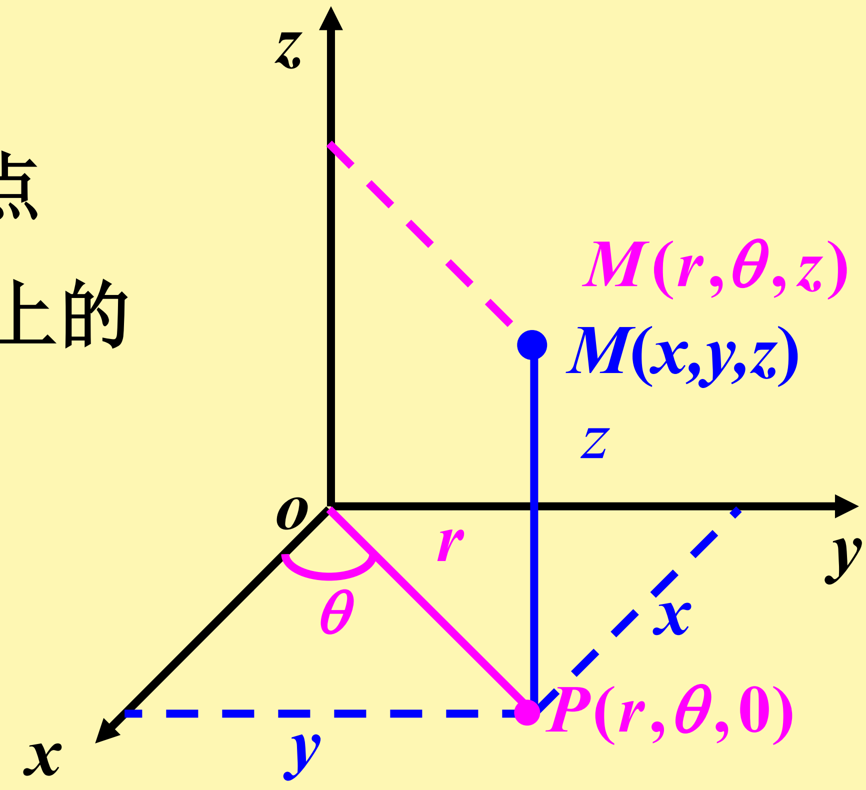
8.3.3 柱面坐标系下的三重积分的计算法

一、柱面坐标

设$M(x,y,z)$为空间内一点

并设点$M(x,y,z)$在xoy面上的投影P的极坐标为$(r,\theta,0)$。

这样的三个数r,θ,z 就叫做点M的柱面坐标。



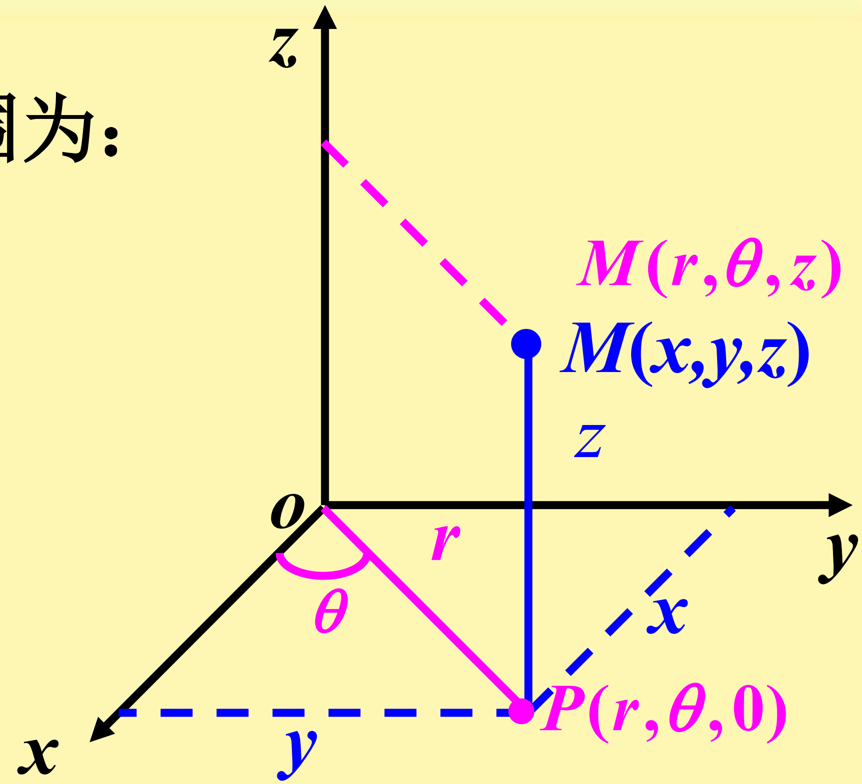
把点先投影后用极坐标表示

①规定r、θ、z的变化范围为:

$0 \le r < +\infty$,

$0 \le \theta \le 2\pi$或 $(-\pi \le \theta \le \pi)$

$-\infty < z < +\infty$



②三组坐标面分别为

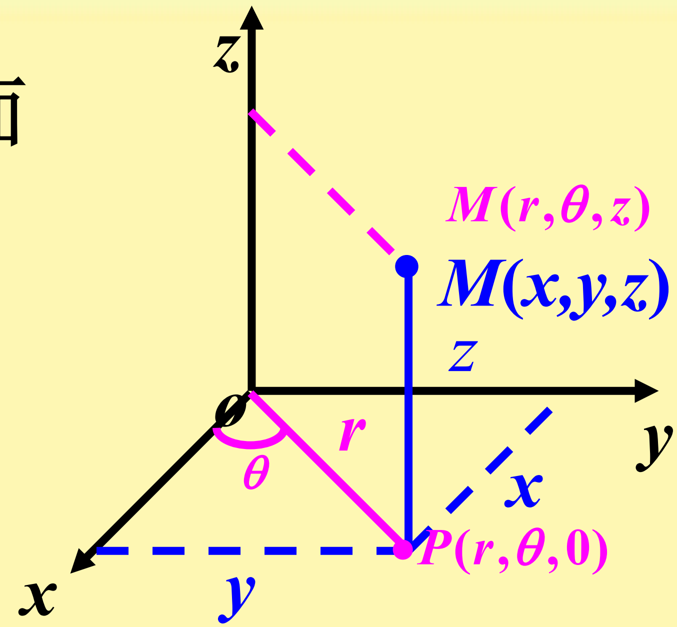
r =常数，即以z 轴为主轴的圆柱面;

θ =常数，即过z 轴的半平面;

z=常数，即与xoy面平行的平面;

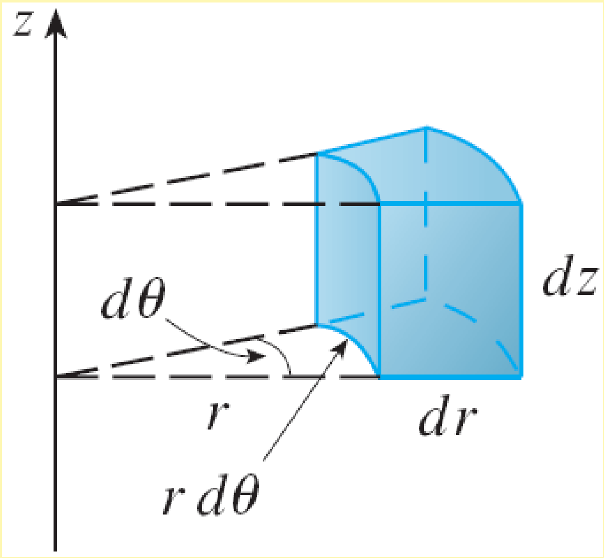
③点M的直角坐标与柱面坐标的关系为：

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$



④柱面坐标系中的体积元素

$$dv = r\,dz\,dr\,d\theta$$



二、柱面坐标中三重积分的形式

坐标变换：$x = r\cos\theta$，$y = r\sin\theta$，$z = z$

$$\iiint_{\Omega} f(x,y,z)dv = \iiint_{\Omega} f(r\cos\theta, r\sin\theta, z)\left|J\right| dz dr d\theta$$

$$= \iiint_{\Omega} f(r\cos\theta, r\sin\theta, z) r \, dz dr d\theta$$

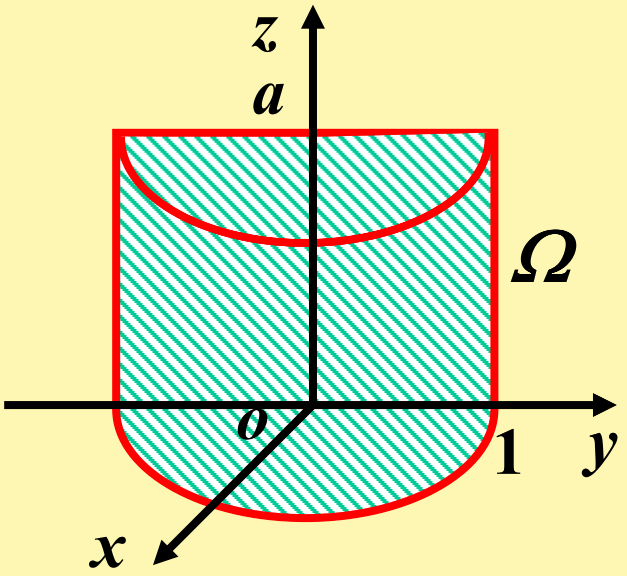
$$J = \frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

例1 将累次积分化为柱面坐标的累次积分, 并求值。

$$\int_0^1 dx\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy\int_0^a z\sqrt{x^2+y^2}\,dz$$

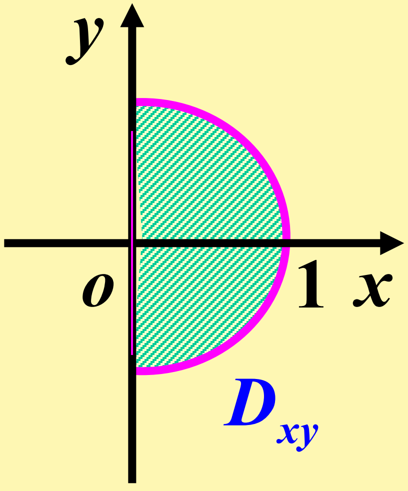
$$=\iiint_\Omega z\sqrt{x^2+y^2}\,dv$$

$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta\int_0^1 dr\int_0^a zr\cdot r\,dz=\frac{\pi}{6}a^2$$



$$\Omega:\begin{cases}0\le z\le a\\ 0\le r\le 1\\ -\dfrac{\pi}{2}\le\theta\le\dfrac{\pi}{2}\end{cases}$$

一般化为先z，再r，最后θ 的 三次积分



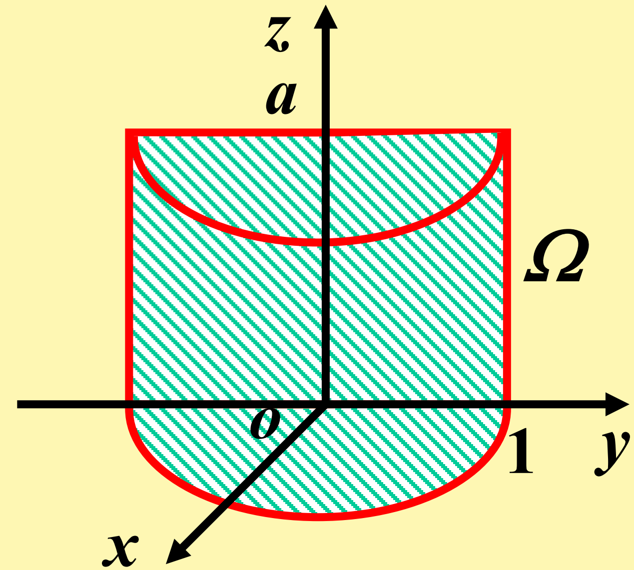
例1 将累次积分化为柱面坐标的累次积分, 并求值。

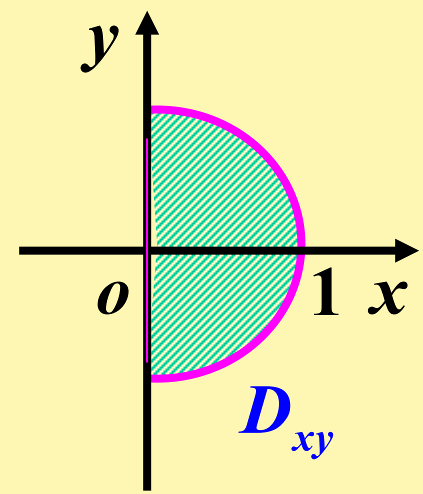
投影法 $\iiint\limits_{\Omega} z\sqrt{x^2+y^2}\,dv$

$$=\iint\limits_{D_{xy}}\left[\int_0^a z\sqrt{x^2+y^2}\,dz\right]dxdy$$

$$=\iint\limits_{D_{xy}}\left[\sqrt{x^2+y^2}\int_0^a zdz\right]dxdy$$

$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\theta\int_0^1 rdr\int_0^a zrdz$$





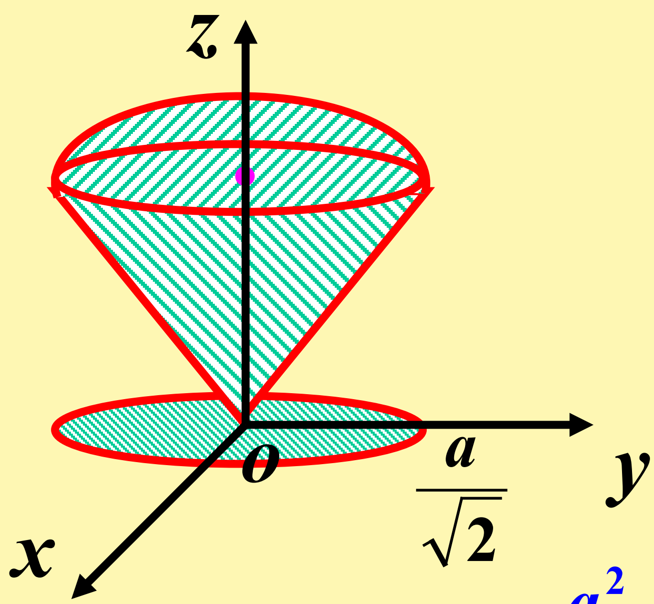
先用投影法后用极坐标表示

对比: $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\theta\int_0^1 dr\int_0^a zr\cdot rdz$

例2　计算 $\iiint\limits_{\Omega} z\,dv$，其中 Ω

由 $z=\sqrt{x^2+y^2}$

和 $z=\sqrt{a^2-x^2-y^2}$ 所围。

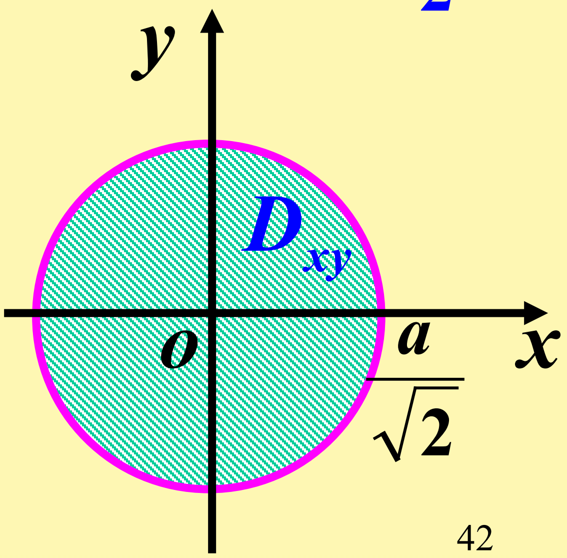


先求圆锥面与球面的交线：

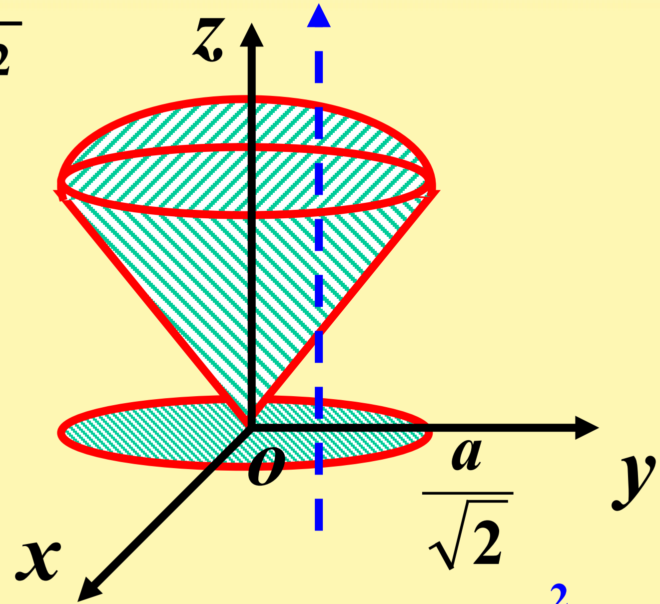
$D_{xy}: x^2+y^2 \le \frac{a^2}{2}$

$$\begin{cases} z=\sqrt{x^2+y^2}, \\ z=\sqrt{a^2-x^2-y^2}, \end{cases} \Rightarrow z^2 = a^2 - z^2$$

$$\Rightarrow z=\frac{\sqrt{2}}{2}a, \quad x^2+y^2=\frac{a^2}{2}$$



$$\Omega:\ \sqrt{x^2+y^2} \le z \le \sqrt{a^2-x^2-y^2}$$

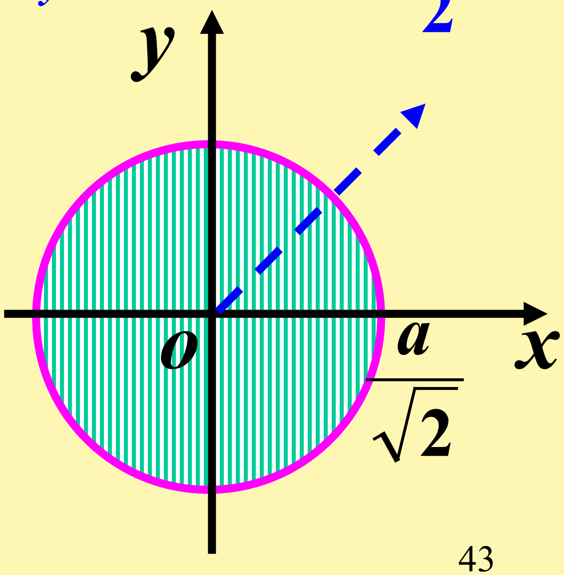


解法一 投影法

$$\iiint_{\Omega} z dv$$

$$= \iint_{D_{xy}} dxdy \int_{\sqrt{x^2+y^2}}^{\sqrt{a^2-x^2-y^2}} zdz$$

$D_{xy}: x^2+y^2 \le \frac{a^2}{2}$

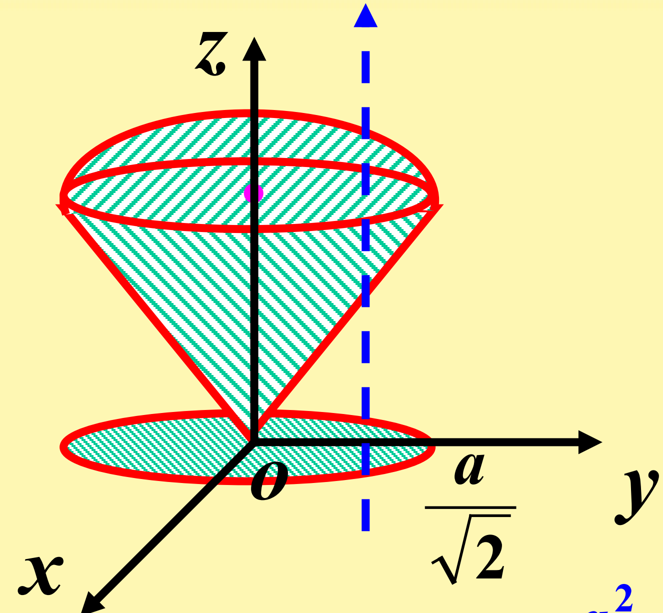


$$= \int_0^{2\pi} d\theta \int_0^{\frac{a}{\sqrt{2}}} rdr \int_{\sqrt{x^2+y^2}}^{\sqrt{a^2-x^2-y^2}} zdz$$

$$= \int_0^{2\pi} d\theta \int_0^{\frac{a}{\sqrt{2}}} rdr \int_r^{\sqrt{a^2-r^2}} zdz = \frac{\pi}{8}a^4$$

$$r=\sqrt{x^2+y^2}\le z\le$$

$$\sqrt{a^2-x^2-y^2}=\sqrt{a^2-r^2}$$



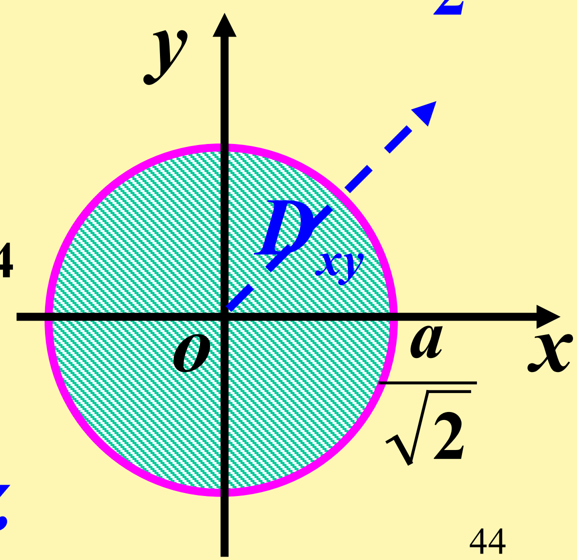
解法二：柱面坐标：

$$\iiint\limits_{\Omega} z\,dv=\iint\limits_{D_{xy}} d\,r\,d\,\theta\int_{r}^{\sqrt{a^2-r^2}} zrdz$$

$$D_{xy}: x^2+y^2\le\frac{a^2}{2}$$

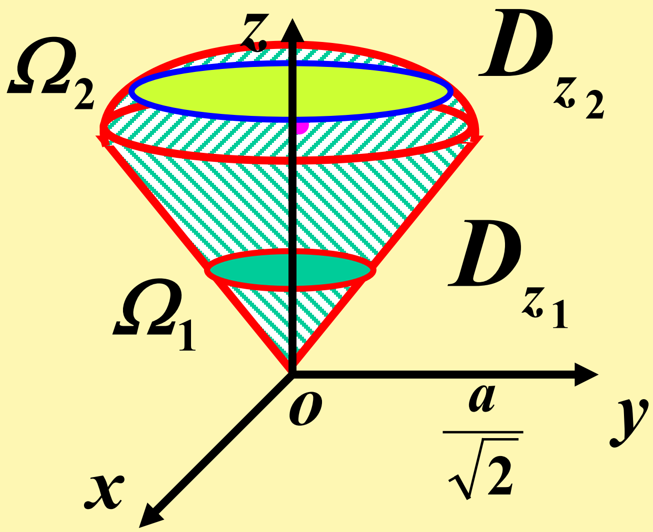
$$=\int_0^{2\pi}d\theta\int_0^{\frac{a}{\sqrt{2}}}dr\int_r^{\sqrt{a^2-r^2}}zrdz$$

$$=\int_0^{2\pi}d\theta\int_0^{\frac{a}{\sqrt{2}}}r\cdot\frac{1}{2}(a^2-2r^2)dr=\frac{\pi}{8}a^4$$



比较：$\int_0^{2\pi}d\theta\int_0^{\frac{a}{\sqrt{2}}}rdr\int_r^{\sqrt{a^2-r^2}}zdz$

解法三 $\iiint\limits_{\Omega} z dv$



Ω 由 $z=\sqrt{x^2+y^2}$ 和

$z=\sqrt{a^2-x^2-y^2}$ 所围。

$$\iiint\limits_{\Omega} z dv = \iiint\limits_{\Omega_1} z dv + \iiint\limits_{\Omega_2} z dv$$

$$= \int_0^{\frac{\sqrt{2}}{2}a} z dz \iint\limits_{D_{z_1}} d\sigma + \int_{\frac{\sqrt{2}}{2}a}^{a} z dz \iint\limits_{D_{z_2}} d\sigma$$

$$= \int_0^{\frac{\sqrt{2}}{2}a} z \cdot \pi z^2 dz + \int_{\frac{\sqrt{2}}{2}a}^{a} z \cdot \pi (a^2 - z^2) dz$$

$$= \frac{\pi}{8} a^4$$

Ω_1 中的截面 D_{z_1}：

$x^2+y^2 \le z^2$

Ω_2 中的截面 D_{z_2}：

$x^2+y^2 \le a^2 - z^2$

分界面：$z=\frac{\sqrt{2}}{2}a$

何时选用柱面坐标计算三重积分？

Ω为旋转体或Ω的边界面中含圆柱面、球面、圆锥面时

投影区域为圆时,被积函数形如 $f(x^2+y^2)$、$f(\frac{y}{x})$等时。

相当于先用投影法，再对投影区域用极坐标表示

一般也可用切片法，有时用切片法更简单

例3 计算 $\iiint\limits_{\Omega} \sqrt{x^2+y^2}\,dv$,

其中$\Omega: x^2+y^2 \le z^2, 1 \le z \le 2$

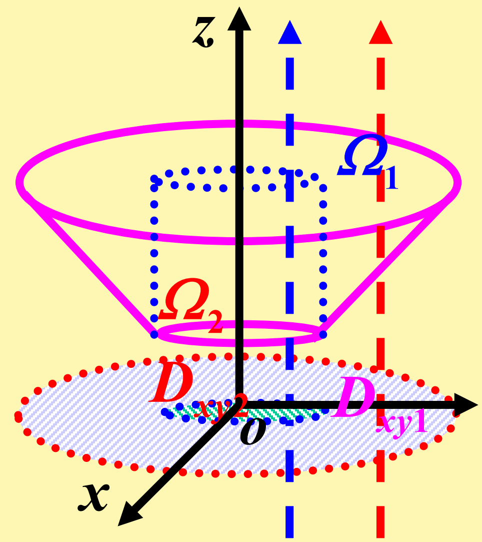
$\Omega_1: \sqrt{x^2+y^2} \le z \le 2$

投影: $1 \le x^2+y^2 \le 4$

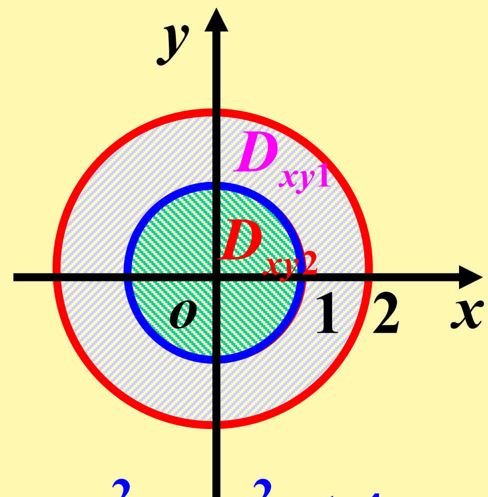
$\Omega_2: 1 \le z \le 2$ 投影: $x^2+y^2 \le 1$

解 $\iiint\limits_{\Omega} \sqrt{x^2+y^2}\,dv$

$$= \iiint\limits_{\Omega_1} \sqrt{x^2+y^2}\,dv + \iiint\limits_{\Omega_2} \sqrt{x^2+y^2}\,dv$$



$\Omega: x^2+y^2 \le z^2, 1 \le z \le 2$



$D_{xy}: x^2+y^2 \le 4$

解 $$\iiint_{\Omega}\sqrt{x^2+y^2}\,dv$$

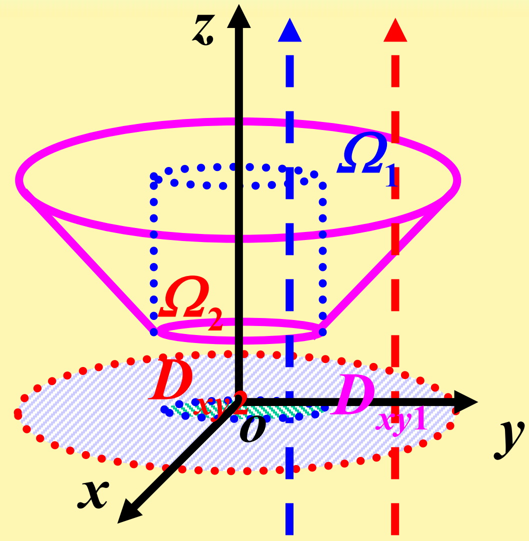
$$=\iiint_{\Omega_1}\sqrt{x^2+y^2}\,dv+\iiint_{\Omega_2}\sqrt{x^2+y^2}\,dv$$

$$=\iint_{D_{xy1}}drd\theta\int_r^2 r\cdot rdz+\iint_{D_{xy2}}drd\theta\int_1^2 r\cdot rdz$$

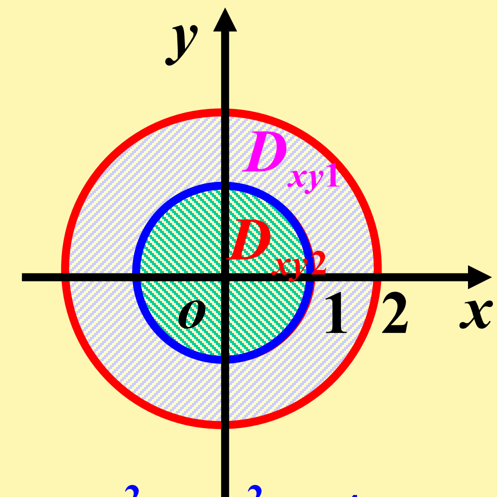
$$=\int_0^{2\pi}d\theta\int_1^2 r\cdot rdr\int_r^2 dz$$

$$+\int_0^{2\pi}d\theta\int_0^1 r\cdot rdr\int_1^2 dz$$

$$=2\pi\left[\int_1^2(2r^2-r^3)dr+\frac{1}{3}\right]=\frac{5\pi}{2}$$



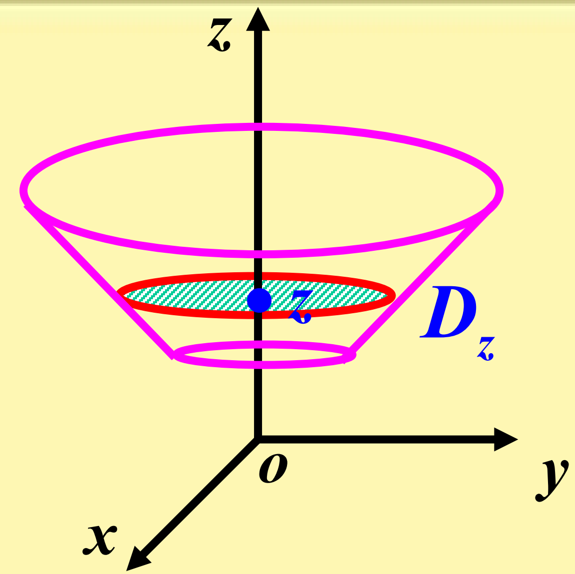
$\Omega: x^2+y^2\le z^2, 1\le z\le 2$



$D_{xy}: x^2+y^2\le 4$

解法二 用截面法

$$\Omega: \ 1 \le z \le 2, (x,y) \in D_z$$



$$\Omega: x^2 + y^2 \le z^2, 1 \le z \le 2$$

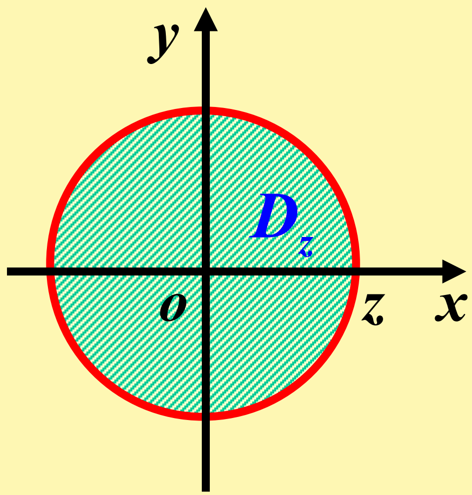
$$\iiint_{\Omega} \sqrt{x^2 + y^2}\, dv$$

$$= \int_1^2 dz \iint_{D_z} \sqrt{x^2 + y^2}\, dxdy$$

$$= \int_1^2 dz \int_0^{2\pi} d\theta \int_0^z r \cdot r dr$$

$$= 2\pi \cdot \int_1^2 \frac{z^3}{3} dz = \frac{2\pi}{3} \cdot \frac{z^4}{4} \Big|_1^2$$

$$= \frac{5\pi}{2}$$



$$D_z: x^2 + y^2 \le z^2$$